

Mean Variance Optimal VWAP Trading*

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Abstract

VWAP is the Volume Weighted Average Price of traded stock over a defined period. It is a metric of trade execution quality used by institutional traders to minimize the execution cost of large trades. A riskless VWAP trading strategy is not possible without knowledge of final market volume. We formulate a mean-variance optimal VWAP strategy by assuming knowledge of final volume and then project this onto the space of strategies accessible to the VWAP trader. The mean variance optimal VWAP trading strategy is the sum of two distinct trading strategies, a minimum variance VWAP hedging strategy and a ‘directional’ price strategy independent of the hedging strategy and market VWAP. It is optimal for large volume VWAP traders to increase the size of the price ‘directional’ trade for additional return.

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1 Introduction

The Volume Weighted Average Price (VWAP) trading strategy is used to minimize the execution costs of large trades in financial markets. This paper formulates continuously updated mean-variance optimal VWAP trading strategies. The optimal VWAP trading strategy is the sum of two conceptually and mathematically different trades, a minimum variance VWAP hedging strategy and an *independent* price ‘directional’ strategy. In addition, we show that the optimal VWAP trading strategies (eqn 16 and eqn 27) are closely associated to the prediction of relative volume and minimize trading price impact by allocating trade volume (liquidity demand) during periods of maximum turnover (liquidity supply).

VWAP is the *total value* of all admissible trades divided by the *total volume* of all admissible trades. If X_i and ΔV_i are the price and volume respectively of the i th trade in the VWAP period $[0, T]$ with N_T total trades, then the VWAP price $\mathcal{V}_T$ calculated at time T is computed as:

$$\mathcal{V}_T = \frac{\text{total traded value}}{\text{total traded volume}} = \frac{\sum_{i=1}^{N_T} X_i \Delta V_i}{\sum_{i=1}^{N_T} \Delta V_i}$$

Traded VWAP price compared to the VWAP price of all admissible trades¹ over the same period was first proposed by Berkowitz, Logue and Noser [2] as a benchmark of execution quality. They argue (page 99) that ‘*a market impact measurement system requires a benchmark price that is an unbiased estimate of prices that could be achieved in any relevant trading period by any randomly selected trader*’. They then define VWAP as an appropriate benchmark that satisfies this criterion. A key feature of VWAP trading is the fragmentation of large trades into smaller trades in order to minimize market impact. The comparative studies of Domowitz and Yegerman [8] and Werner [35] show that the execution costs of VWAP are comparable to, or lower than, other block (large) trading strategies.

For any institutional (large) trader the cost of trading is as important as the decision to trade. The cost of trading has three components: fees

¹Not all trades are accepted as admissible in a VWAP calculation. Admissible trades are determined by market convention and are generally on-market trades. Off-market trades and crossings are generally excluded from the VWAP calculation because these trades are often priced away from the current market and represent volume in which a ‘*randomly selected trader*’ [2] cannot participate.

and commissions (generally fixed); the cost of execution; and opportunity cost. Execution cost (also known as ‘implementation shortfall’, Peróld [25]) is the difference in stock price between the time the decision to trade is taken and the eventual traded price. The major component of the execution cost is market impact cost, which is the adverse price movement due to the execution of a large trade on the market. Market impact cost results from the liquidity demand of large trades and information signalled to other market participants by the presence of a large trade in the market. Market impact cost can be reduced by fragmenting a large trade into smaller trades and spreading instantaneous liquidity demand over a longer trading period. Bertsimas and Lo [3] have formulated an optimal trade fragmentation schedule using a linear liquidity supply curve to model market impact.

Opportunity cost is foregone profit on volume that was not executed because of adverse price movement subsequent to the decision to trade. Opportunity cost increases with time and since market impact cost decreases with time then total (execution + opportunity) transaction costs are convex in time. An optimal trading schedule based on a convex total transaction cost curve was formulated by Almgren and Chriss [1] using a market impact model that differentiates temporary and permanent adverse price movement due to liquidity demand.

A non-adaptive optimal VWAP trading strategy fixed at the beginning of the VWAP period $t = 0$ ($\mathcal{F}_0$ adapted) has been formulated by Konishi [19] for price modeled as a (martingale) Wiener process. We extend Konishi’s formulation to a continuously updated dynamic mean-variance optimal VWAP strategy using the information available at time t ($\mathcal{F}_t$ adapted) with price modeled as a continuous semimartingale. In addition, we show that the optimal VWAP trading strategies (eqn 16 and eqn 27) are closely associated to the prediction of relative volume and minimize trading price impact by allocating trade volume (liquidity demand) during periods of maximum turnover (liquidity supply).

VWAP is naturally defined using relative volume Y rather than cumulative volume V . The relative volume process Y is defined as market cumulative volume V_t divided by market total final volume $Y_t = V_t/V_T$ and is adapted to a filtration $\mathcal{G}$ which is the VWAP trader accessible (observed) filtration $\mathcal{F}$ enlarged by knowledge of final volume $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(V_T)$. The VWAP price $\mathcal{V}_T$ (eqn 1) at time T can be formulated as an Itô integral under $\mathcal{G}$ with a continuous price X integrator and with a $\mathcal{G}$ -predictable relative volume process integrand $\xi_t^{\mathcal{V}, \mathcal{G}} = 1 - Y_{t-}$. No zero risk $\mathcal{F}$ adapted (trader

accessible) VWAP trading strategy exists (lemma 3.8). The continuous time formulation of VWAP price $\mathcal{V}_T$ is:

$$\mathcal{V}_T = X_0 + \int_0^T \xi^{\mathcal{V},\mathcal{G}} dX \quad (1)$$

The $\mathcal{F}$ adapted mean square optimal VWAP trading strategy is derived:

$$\xi^{\mathcal{V},\mathcal{F}} = \min_{\xi^{\mathcal{F}}} \mathbb{E} \left[\left(\int_0^T (\xi^{\mathcal{F}} - \xi^{\mathcal{V},\mathcal{G}}) dX \right)^2 \right] \quad (2)$$

This is then extended by deriving the $\mathcal{F}$ adapted Markowitz mean-variance optimal trading strategy:

$$\xi^{\mathcal{V},\lambda,\mathcal{F}} = \max_{\xi^{\mathcal{F}}} \left[\mathbb{E} \left[\int_0^T (\xi^{\mathcal{F}} - \xi^{\mathcal{V},\mathcal{G}}) dX \right] - \frac{\lambda}{2} \text{Var} \left[\int_0^T (\xi^{\mathcal{F}} - \xi^{\mathcal{V},\mathcal{G}}) dX \right] \right] \quad (3)$$

The optimal mean variance VWAP trading strategy $\xi^{\mathcal{V},\lambda,\mathcal{F}}$ is the sum of two distinct trading strategies, a minimum variance VWAP hedging strategy $\lambda \rightarrow \infty$; $\xi^{\mathcal{V},\infty,\mathcal{F}}$ and a ‘directional’ price strategy ξ_0^1 independent of the hedging strategy and market VWAP. The ‘directional’ price strategy uses the properties of the Variance Optimal Martingale Measure (definition 3.2) to determine additional variance and expected return (section 3.3.2).

$$\xi^{\mathcal{V},\lambda,\mathcal{F}} = \xi^{\mathcal{V},\infty,\mathcal{F}} + \frac{\xi_0^{1,\mathcal{F}}}{\lambda} \quad (4)$$

This is a general result, see McCulloch [22] for details. Any mean-variance hedging strategy (including VWAP) can be formulated as the sum of two conceptually and mathematically different trades, a minimum variance hedging strategy and an *independent* price ‘directional’ strategy. The price ‘directional’ strategy is only dependent on the stochastic characteristics of the price process X . Furthermore, the additional variance of the optimal mean-variance solution is a direct sum of the minimum variance hedging strategy variance and the variance of the price ‘direction’ strategy. Similarly, the additional expectation of the optimal mean-variance solution is a direct sum of

the minimum variance hedging strategy expectation and the expectation of the price ‘direction’ strategy (eqns 24 and 25).

Define β as the proportion of total market volume V_T that a VWAP trader intends to trade. A VWAP trader with a βV_T trade size has a risk aversion parameter of $(1 - \beta)\lambda$ (section 3.3.3). Where λ is the trader risk aversion parameter for a small VWAP ($\beta \ll 1$) trade size. Thus it is optimal for large $0 < \beta < 1$ mean-variance VWAP traders to increase the size of the price ‘directional’ trade for additional return. This is an intuitive result, traders who control a significant proportion of total volume can influence final VWAP price. In the limit, a VWAP trader who controls all volume completely determines the VWAP price.

$$\xi^{\nu, \lambda, \mathcal{F}} = \xi^{\nu, \infty, \mathcal{F}} + \frac{\xi_0^{1, \mathcal{F}}}{(1 - \beta)\lambda} \quad (5)$$

Section 3.4 introduces simplifying assumptions for ‘practical’ VWAP optimal trading. Specifically, we obtain a solution if the Variance Optimal Martingale Measure (VOMM) and Minimal Martingale Measure (MMM) coincide. We also obtain a solution if final volume and price are assumed independent. Finally, we have an optimal VWAP example that assumes the price process is a Geometric Brownian Motion.

The mean-variance optimal VWAP trading strategies developed in this paper are continuous, assume frictionless markets (no transaction costs) and have no constraint on trading direction. These important practical considerations are held over for future research.

2 Modelling VWAP

2.1 Price and Volume

Assumption 2.1.

- (i) *The stochastic environment of the VWAP asset is endowed with a stochastic basis $(\Omega, \mathcal{F}_T, \mathcal{F}, \mathbb{P})$ with a standard (right continuous and complete) filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$. The filtration $\mathcal{F}$ contains all VWAP trader accessible (observable) information about the VWAP asset. In addition, the price process X is assumed to be a $\mathcal{F}$ continuous square integrable semimartingale $X \in \mathcal{S}^2(\mathbb{P}, \mathcal{F})$ and the cumulative volume process $V \in \mathcal{S}^2(\mathbb{P}, \mathcal{F})$ is also a $\mathcal{F}$ square integrable semimartingale.*
- (ii) *VWAP trading is normally conducted on single financial market assets. Therefore the exposition will assume univariate price X and cumulative volume V processes.*

2.1.1 Final Volume, Relative Volume and the Initially Enlarged Filtration

Final volume is a square integrable positive integer valued random variable, $V_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ (abbreviated $V_T \in L^2(\mathbb{P})$). A finite set of possible final volumes exists $V_T \in S_V$, where the minimum element is the VWAP final volume $v_T = \min[v \in S_V]$ (no other market participants trade in the VWAP period) and the maximum element is bounded, $\max[v \in S_V] < \infty$.

The filtration $\mathcal{G}$ is the VWAP trader accessible filtration $\mathcal{F}$ initially enlarged by the sigma algebra generated by final volume $\sigma(V_T)$. Since V_T is $\mathcal{F}_T$ adapted then $\mathcal{F}_T = \mathcal{G}_T$ and it is a standard filtration under $\mathbb{P}$. The enlarged filtration $\mathcal{G}$ is formally defined as:

$$\mathcal{G}_t = \bigcap_{\epsilon > 0} (\mathcal{F}_{t+\epsilon} \vee \sigma(V_T))$$

Remark 2.2. *Since the sigma algebra $\sigma(V_T)$ of final cumulative volume is finitely large, the $\mathcal{F}$ semimartingale price process X is also a $\mathcal{G}$ semimartingale (Jacod [16]).*

Assumption 2.3. *The price process X is a strictly positive continuous square integrable semimartingale with respect to the enlarged filtration $\mathcal{G}$, $X \in \mathcal{S}^2(\mathbb{P}, \mathcal{G})$.*

The relative volume process is progressive cumulative volume divided by the final cumulative volume.

$$Y_t = \frac{V_t}{V_T} \quad t \in [0, T] \quad (6)$$

Relative volume is zero at time zero $Y_0 = 0$, non-decreasing and has a value of 1 at time T , $Y_T = 1$. It is measurable with respect to the enlarged filtration $\mathcal{G}$ and is a square integrable semimartingale $Y \in \mathcal{S}^2(\mathbb{P}, \mathcal{G})$.

2.2 A Continuous Time Model of VWAP

One the reasons for the popularity of VWAP as a measure of order execution quality is the simplicity of its definition - the *total value* of all admissible trades divided by the *total volume* of all admissible trades. If X_i and ΔV_i are the price and volume respectively of the i th trade in the VWAP period $[0, T]$ with N_T total trades, then the VWAP price $\mathcal{V}_T$ calculated at time T is readily computed as:

$$\mathcal{V}_T = \frac{\text{total traded value}}{\text{total traded volume}} = \frac{\sum_{i=1}^{N_T} X_i \Delta V_i}{\sum_{i=1}^{N_T} \Delta V_i}$$

Alternatively, the definition of VWAP can be written in continuous time notation. The price process X is a $\mathcal{G}$ semimartingale (remark 2.2), therefore VWAP can be formulated as a $\mathcal{G}$ adapted Itô integral.

$$\mathcal{V}_T = \frac{\text{total traded value}}{\text{total traded volume}} = \frac{1}{V_T} \int_0^T X_- dV \quad (7)$$

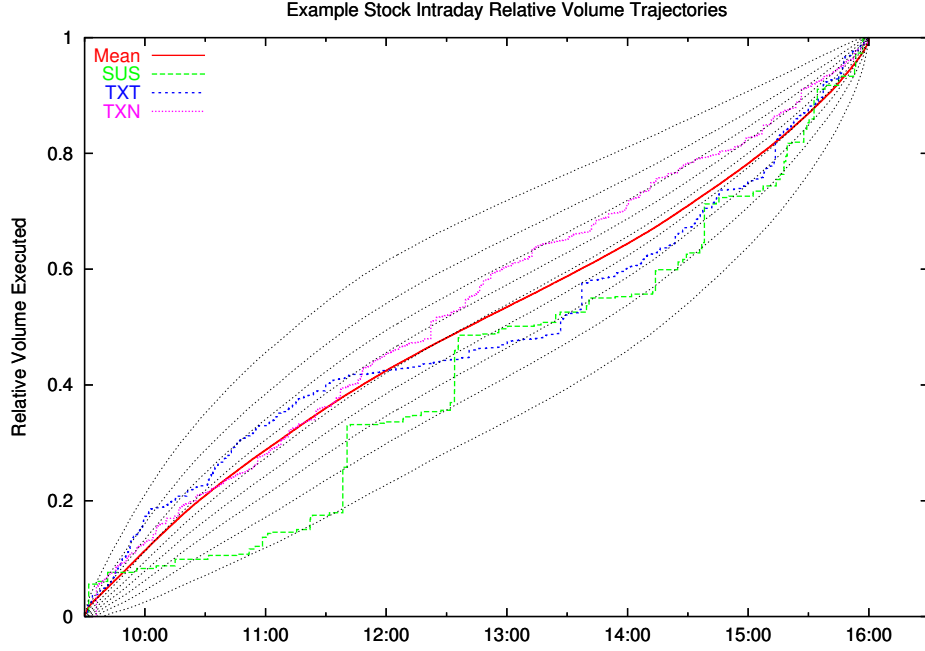


Figure 1: This graph shows an example of relative volume trajectories Y_t for 3 stocks representing low, medium and high turnover stocks. The red line is the expected relative volume $\mathbb{E}[Y_t]$ for all stocks trading more than 50 trades a day on the NYSE. This expectation is the integral of the ‘U’ shaped trading intensity ($d\mathbb{E}[Y_t]/dt$) found on all major stock markets. *SUS* is *Storage USA*, *TXT* is *Textron Incorporated* and *TXN* is *Texas Instruments*; on 2 July 2001 these stocks recorded 101, 946 and 2183 trades correspondingly.

Examining the integral above, it is readily seen that it relates to the relative volume process $Y_t = V_t/V_T$. This is a key insight; *VWAP is naturally defined using relative volume Y_t rather than actual volume V_t .*

$$\mathcal{V}_T = \frac{1}{V_T} \int_0^T X_- dV = \int_0^T X_- dY$$

Integration by parts gives:

$$\mathcal{V}_T = X_T Y_T - X_0 Y_0 - \int_0^T Y_- dX - [X, Y]_T$$

Since the relative volume process Y is monotonically non-decreasing and therefore of finite variation and X is $\mathcal{G}$ continuous by assumption 2.3, the

quadratic covariation term is zero $[X, Y] = 0$. Also $Y_0 = 0$ and $Y_T = 1$ by definition, so the integration by parts equation simplifies to:

$$\mathcal{V}_T = X_T - \int_0^T Y_- dX = X_0 + \int_0^T (1 - Y_-) dX$$

Defining the $\mathcal{G}$ predictable integrand:

$$\xi_t^{\mathcal{V}, \mathcal{G}} = 1 - Y_{t-} = 1 - \frac{V_{t-}}{V_T} \quad (8)$$

Using this definition, VWAP has the following Itô integral representation:

$$\mathcal{V}_T = X_0 + \int_0^T \xi^{\mathcal{V}, \mathcal{G}} dX$$

Remark 2.4. *Note that unlike a conventional trading strategy where the constant term represents initial trading capital, the X_0 term above (eqn 1) is initial stock price and cannot be specified by a VWAP trader. Consequently, this constant term plays no role in optimizing VWAP trading strategies.*

Since the price process is square integrable by assumption and the relative volume process is bounded by 1 ($0 \leq Y_t \leq 1$), the VWAP random variable $\mathcal{V}_T$ is square integrable $\mathcal{V}_T \in L^2(\mathbb{P})$.

3 Optimal VWAP Trading Strategies

It is shown (lemma 3.8) that VWAP is not *attainable* (Schweizer [32]) in the VWAP trader accessible filtration $\mathcal{F}$ and any feasible $\mathcal{F}$ adapted trading strategy is risky. Therefore the objective is to specify a feasible minimum risk $\mathcal{F}$ adapted VWAP strategy and extend this to a Markowitz mean-variance optimal strategy. This is an application of the theory of the optimal quadratic projection of random variables $H \in L^2(\mathbb{P})$ onto a subspace of Itô integrals $(\xi \cdot X)_T \in L^2(\mathbb{P})$. The extensive quadratic hedging literature includes Chunli and Karatzas [15], Schweizer [32] [33], Delbaen and Schachermayer [7], Rheinländer and Schweizer [29], Pham and Rheinländer and Schweizer [26], Gouriéroux and Laurent and Pham [13] and Rheinländer [28].

Optimal VWAP trading belongs to the class of hedging problems where the accessible filtration $\mathcal{F}$ is smaller than the filtration $\mathcal{G}$ required for hedge replication to be attainable. This has been studied by Schweizer [30] for martingale X and has also been studied by Föllmer and Schweizer [10], who gave a solution that was local error minimizing for semimartingale X . Møller [23] [24] and Schweizer [34] extended this to a Markowitz mean-variance optimal solution for semimartingale X .

This section formulates a minimum mean-square (L^2) VWAP strategy for martingale X , extends this to a mean-square (L^2) optimal strategy for semimartingale X (eqn 2) and finally develops a Markowitz mean-variance optimal VWAP strategy ($\lambda \geq 0$) for semimartingale X (eqn 3).

3.1 Existence and Closure Conditions

Finding the optimal $\mathcal{F}$ adapted VWAP strategy $\xi^{\mathcal{F}}$ is essentially a Hilbert space projection from the space of stochastic integrals with $\mathcal{G}$ adapted integrands to the space of stochastic integrals with $\mathcal{F}$ adapted integrands. Therefore the closure of the subspace of stochastic integrals with $\mathcal{F}$ adapted integrands is a necessary property and is detailed below. Similarly, the conditions for the existence of Equivalent Martingale Measures, and in particular the Variance Optimal Martingale Measure, on the Hilbert spaces of stochastic integrals is detailed below. These are standard results from the quadratic hedging literature.

3.1.1 Existence of Equivalent Local Martingale Measures

The Itô integral with integrand ξ is notated:

$$G_T(\xi) = \int_0^T \xi dX$$

Let $L(X, \mathcal{G})$ denote the space of all X -integrable $\mathcal{G}$ predictable processes. The set of admissible $\mathcal{G}$ predictable integrands where the resulting integration is a square integrable semimartingale is:

$$\Theta(\mathcal{G}) = \{ \xi^{\mathcal{G}} \in L(X, \mathcal{G}) \mid (\xi^{\mathcal{G}} \cdot X)_t \in \mathcal{S}^2(\mathbb{P}, \mathcal{G}) \}$$

Note that $\xi^{\mathcal{V}, \mathcal{G}} \in \Theta(\mathcal{G})$ by definition. The set of signed local martingale measures such that any $g \in G_T(\Theta(\mathcal{G}))$ is a local martingale (these definitions from Delbaen and Schachermayer [7]) are defined:

$$\mathcal{D}^s(\mathcal{G}) = \{ Z^{\mathcal{G}} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \mid \forall g \in G_T(\Theta(\mathcal{G})), \mathbb{E}[Z^{\mathcal{G}} g] = 0, \mathbb{E}[Z^{\mathcal{G}}] = 1 \}$$

$$\mathcal{M}^s(\mathcal{G}) = \{ \mathbb{Q}^{\mathcal{G}} \mid d\mathbb{Q}^{\mathcal{G}}/d\mathbb{P} = Z^{\mathcal{G}}, Z^{\mathcal{G}} \in \mathcal{D}^s(\mathcal{G}) \}$$

From lemma 2.1 in Delbaen and Schachermayer [7] the set of *signed local martingale measures* $\mathcal{D}^s(\mathcal{G}) \neq \emptyset$ is non-empty if the constant 1 is not contained in $G_T(\Theta(\mathcal{G}))$.

Assumption 3.1. $1 \notin G_T(\Theta(\mathcal{G}))$ and therefore $\mathcal{D}^s(\mathcal{G}) \neq \emptyset$.

The set of equivalent local martingale measures is the subset of signed local martingale measures that are strictly positive and therefore probability measures.

$$\mathcal{D}^e(\mathcal{G}) = \{ Z^{\mathcal{G}} \in \mathcal{D}^s(\mathcal{G}) \mid Z^{\mathcal{G}} > 0 \}$$

$$\mathcal{M}^e(\mathcal{G}) = \{ \mathbb{Q}^{\mathcal{G}} \mid d\mathbb{Q}^{\mathcal{G}}/d\mathbb{P} = Z^{\mathcal{G}}, Z^{\mathcal{G}} \in \mathcal{D}^e(\mathcal{G}) \}$$

The sets of signed local martingale measures and equivalent local martingale measures are equal ($\mathcal{D}^e(\mathcal{G}) = \mathcal{D}^s(\mathcal{G})$, $\mathcal{M}^e(\mathcal{G}) = \mathcal{M}^s(\mathcal{G})$) if price process X is $\mathcal{G}$ continuous (Delbaen and Schachermayer [7]). Therefore by assumption 2.3 $\mathcal{M}^e(\mathcal{G}) \neq \emptyset$.

The proper subset (lemma 3.8) $\Theta(\mathcal{F}) \subset \Theta(\mathcal{G})$ is defined where all the elements are $\mathcal{F}$ adapted. Let $L(X, \mathcal{F})$ denote the space of all X -integrable $\mathcal{F}$ predictable processes:

$$\Theta(\mathcal{F}) = \{ \xi^{\mathcal{F}} \in L(X, \mathcal{F}) \mid (\xi^{\mathcal{F}} \cdot X)_t \in \mathcal{S}^2(\mathbb{P}, \mathcal{F}) \}$$

Finding the optimal mean square adapted strategy $\xi^{\mathcal{F}}$ is a Hilbert space projection from $L^2(\mathbb{P})$ onto $G_T(\Theta(\mathcal{F}))$. Therefore the closure of $G_T(\Theta(\mathcal{F}))$ in $L^2(\mathbb{P})$ is a necessary property. The set of equivalent local martingale measures such that any $f \in G_T(\Theta(\mathcal{F}))$ is a local martingale is defined:

$$\mathcal{D}^s(\mathcal{F}) = \{ Z^{\mathcal{F}} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \mid \forall f \in G_T(\Theta(\mathcal{F})), \mathbb{E}[Z^{\mathcal{F}} f] = 0, \mathbb{E}[Z^{\mathcal{F}}] = 1 \}$$

From the $\mathcal{F}$ continuity of X (assumption 2.1) the sets of signed local martingale measures and equivalent local martingale measures are equal:

$$\mathcal{D}^e(\mathcal{F}) = \mathcal{D}^s(\mathcal{F}) \tag{9}$$

$$\mathcal{M}^e(\mathcal{F}) = \{ \mathbb{Q}^{\mathcal{F}} \mid d\mathbb{Q}^{\mathcal{F}}/d\mathbb{P} = Z^{\mathcal{F}}, Z^{\mathcal{F}} \in \mathcal{D}^e(\mathcal{F}) \}$$

Since $G_T(\Theta(\mathcal{F})) \subset G_T(\Theta(\mathcal{G}))$ it is intuitive and true that $\mathcal{D}^e(\mathcal{G}) \subseteq \mathcal{D}^e(\mathcal{F})$ and $\mathcal{M}^e(\mathcal{G}) \subseteq \mathcal{M}^e(\mathcal{F})$ and therefore from assumption 3.1 $\mathcal{M}^e(\mathcal{F}) \neq \emptyset$.

3.1.2 Closure of the Itô Integral Subspaces

Definition 3.2. The $\mathcal{F}$ Variance Optimal Martingale Measure (VOMM) is defined as the equivalent local martingale measure $\tilde{\mathbb{Q}}^{\mathcal{F}} \in \mathcal{M}^e(\mathcal{F})$ such that the associated density $\tilde{Z}^{\mathcal{F}} \in \mathcal{D}^e(\mathcal{F})$ has the minimum $L^2(\mathbb{P})$ norm:

$$\frac{d\tilde{\mathbb{Q}}^{\mathcal{F}}}{d\mathbb{P}} = \tilde{Z}^{\mathcal{F}} = \min_{Z^{\mathcal{F}} \in \mathcal{D}^e(\mathcal{F})} \|Z^{\mathcal{F}}\|_{L^2(\mathbb{P})} = \min_{Z^{\mathcal{F}} \in \mathcal{D}^e(\mathcal{F})} \text{Var}[Z^{\mathcal{F}}]$$

Definition 3.3. A uniformly integrable strictly positive $(\mathbb{P}, \mathcal{F})$ martingale $Z > 0$ satisfies the $\mathcal{F}$ reverse Hölder condition, denoted $Z \in \mathcal{R}_p(\mathbb{P}, \mathcal{F})$, if there is a constant $C < \infty$ such that for every stopping time $0 \leq \tau \leq T$ the following relationship exists:

$$\mathbb{E} \left[\left(\frac{Z}{Z_\tau} \right)^p \middle| \mathcal{F}_\tau \right] \leq C \quad p \in (1, \infty)$$

Assumption 3.4. The VOMM density $\tilde{Z}^{\mathcal{F}}$ satisfies the $\mathcal{F}$ reverse Hölder inequality, $\tilde{Z}^{\mathcal{F}} \in \mathcal{R}_2(\mathbb{P}, \mathcal{F})$.

If the $\mathcal{F}$ VOMM density satisfies the $\mathcal{F}$ reverse Hölder inequality $\tilde{Z}^{\mathcal{F}} \in \mathcal{M}^e(\mathcal{F}) \cap \mathcal{R}_2(\mathbb{P}, \mathcal{F})$, the closure of $G_T(\Theta(\mathcal{F}))$ in $L^2(\mathbb{P})$ was proved by Delbaen, Monat, Schachermayer, Schweizer and Stricker [6] (theorem 4.1).

3.2 L^2 Optimal Strategies

3.2.1 An L^2 Optimal Strategy for Martingale X

Let $\mathcal{P}\mathcal{G}$ denote the predictable sigma field of $\mathcal{G}$ and $\mathcal{B}[0, T]$ the Borel algebra on the interval $[0, T]$. The sigma finite Doléans measure $\nu_{\mathbb{Q}}$ and associated product space can be defined using continuous square integrable $\mathcal{G}$

adapted local martingale X under any equivalent local martingale measure $\mathbb{Q} \in \mathcal{M}^e(\mathcal{G})$:

$$(\Omega \times [0, T], \mathcal{P}\mathcal{G} \otimes \mathcal{B}[0, T], \nu_{\mathbb{Q}})$$

$$\forall A \in \mathcal{P}\mathcal{G} \otimes \mathcal{B}[0, T]; \quad \nu_{\mathbb{Q}}(A) = \mathbb{E}^{\mathbb{Q}} \left[\int_0^T \mathbf{1}_A d\langle X \rangle \right] = \int_{\Omega} \int_0^T \mathbf{1}_A(t, \omega) d\langle X \rangle_t(\omega) d\mathbb{Q}(\omega).$$

Any predictable process $\xi^{\mathcal{G}}$ adapted to $\mathcal{P}\mathcal{G}$ is an element of the associated Hilbert space $\xi^{\mathcal{G}} \in L^2(\Omega \times [0, T], \mathcal{P}\mathcal{G} \otimes \mathcal{B}[0, T], \nu_{\mathbb{Q}})$ (abbreviated $L^2(X, \mathbb{Q})$) with norm:

$$\|\xi^{\mathcal{G}}\|_{L^2(X, \mathbb{Q})}^2 = \mathbb{E}^{\mathbb{Q}} \left[\int_0^T (\xi^{\mathcal{G}})^2 d\langle X \rangle \right]$$

It is a standard result (Kunita and Watanabe [20]) that $L^2(X, \mathbb{Q})$ is isometric to the corresponding norm on $L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$ and the isometry mapping is the Itô integral $I_X : \xi \mapsto (\xi \cdot X)_T$.

$$\|\xi^{\mathcal{G}}\|_{L^2(X, \mathbb{Q})}^2 = \mathbb{E}^{\mathbb{Q}} \left[\int_0^T (\xi^{\mathcal{G}})^2 d\langle X \rangle \right] = \|I_X(\xi^{\mathcal{G}})\|_{L^2(\mathbb{Q})}^2$$

Define the standard predictable σ -subfield $\mathcal{H} \subseteq \mathcal{P}\mathcal{G}$, we assert that $L^2(\Omega \times [0, T], \mathcal{H} \otimes \mathcal{B}[0, T], \nu_{\mathbb{Q}})$ is a Hilbert subspace². The projection operator onto this subspace is denoted:

$$\xi^{\mathcal{H}} = P^{\mathcal{H}}(\xi^{\mathcal{G}})$$

The lemma below specifies the explicit time indexed process $\xi_t^{\mathcal{H}}$ that results from projection onto the subspace.

²See the closure conditions for the isometric L^2 spaces in section 3.1.

Lemma 3.5. Define $\mathcal{H} \subseteq \mathcal{P}\mathcal{G}$ as a standard predictable σ -subfield. For any $\mathcal{P}\mathcal{G}$ adapted process $\xi^\mathcal{G} \in (\Omega \times [0, T], \mathcal{P}\mathcal{G} \otimes \mathcal{B}[0, T], \nu_\mathbb{Q})$ the projection of $\xi^\mathcal{G}$ onto the subspace $L^2(\Omega \times [0, T], \mathcal{H} \otimes \mathcal{B}[0, T], \nu_\mathbb{Q})$ is formulated explicitly as:

$$\xi_t^\mathcal{H} = P^\mathcal{H}(\xi_t^\mathcal{G}) = \frac{\mathbb{E}^\mathbb{Q}[\xi_t^\mathcal{G} d\langle X \rangle_t | \mathcal{H}_t]}{\mathbb{E}^\mathbb{Q}[d\langle X \rangle_t | \mathcal{H}_t]} \quad \forall t \in [0, T] \quad (10)$$

Proof. It is straightforward to show that $P^\mathcal{H}$ is bounded and idempotent. Using Fubini's theorem and the definition of the Hilbert space norm we show that $P^\mathcal{H}$ is self-adjoint. For any $\eta \in (\Omega \times [0, T], \mathcal{P}\mathcal{G} \otimes \mathcal{B}[0, T], \nu_\mathbb{Q})$ the inner product (denoted $\langle \cdot, \cdot \rangle_{L^2(X, \mathbb{Q})}$) is:

$$\begin{aligned} \langle P^\mathcal{H}(\xi^\mathcal{G}), \eta \rangle_{L^2(X, \mathbb{Q})} &= \int_\Omega \int_0^T P^\mathcal{H}(\xi_t^\mathcal{G}) \eta_t d\langle X \rangle_t d\mathbb{Q}^\mathcal{G} \\ &= \int_0^T \mathbb{E}^\mathbb{Q} \left[\frac{\mathbb{E}^\mathbb{Q}[\xi_t^\mathcal{G} d\langle X \rangle_t | \mathcal{H}_t]}{\mathbb{E}^\mathbb{Q}[d\langle X \rangle_t | \mathcal{H}_t]} \eta_t d\langle X \rangle_t \right] \\ &= \int_0^T \mathbb{E}^\mathbb{Q} \left[\mathbb{E}^\mathbb{Q} \left[\frac{\mathbb{E}^\mathbb{Q}[\xi_t^\mathcal{G} d\langle X \rangle_t | \mathcal{H}_t]}{\mathbb{E}^\mathbb{Q}[d\langle X \rangle_t | \mathcal{H}_t]} \eta_t d\langle X \rangle_t \middle| \mathcal{H}_t \right] \right] \\ &= \int_0^T \mathbb{E}^\mathbb{Q} \left[\frac{\mathbb{E}^\mathbb{Q}[\xi_t^\mathcal{G} d\langle X \rangle_t | \mathcal{H}_t]}{\mathbb{E}^\mathbb{Q}[d\langle X \rangle_t | \mathcal{H}_t]} \mathbb{E}^\mathbb{Q}[\eta_t d\langle X \rangle_t | \mathcal{H}_t] \right] \end{aligned}$$

Using the same manipulations we obtain:

$$\langle \xi^\mathcal{G}, P^\mathcal{H}(\eta) \rangle_{L^2(X, \mathbb{Q})} = \int_0^T \mathbb{E}^\mathbb{Q} \left[\frac{\mathbb{E}^\mathbb{Q}[\eta_t d\langle X \rangle_t | \mathcal{H}_t]}{\mathbb{E}^\mathbb{Q}[d\langle X \rangle_t | \mathcal{H}_t]} \mathbb{E}^\mathbb{Q}[\xi_t^\mathcal{G} d\langle X \rangle_t | \mathcal{H}_t] \right]$$

Thus $P^\mathcal{H}$ is bounded, idempotent and self-adjoint and is an orthogonal projection. □

Since $P^{\mathcal{H}}$ is an orthogonal projection, the predictable $\mathcal{H}$ adapted process that minimizes the norm $\|\xi^{\mathcal{G}} - \xi^{\mathcal{H}}\|_{L^2(X, \mathbb{Q})}$ is $\xi^{\mathcal{H}} = P^{\mathcal{H}}(\xi^{\mathcal{G}})$. It is immediate from the isometry [20] that $\xi^{\mathcal{H}}$ also minimizes $\|(\xi^{\mathcal{G}} \cdot X)_T - (\xi^{\mathcal{H}} \cdot X)_T\|_{L^2(\mathbb{Q})}$.

Remark 3.6. *The mean square optimal $\mathcal{H}$ adapted trading strategy for $(\mathbb{Q}, \mathcal{G})$ martingale X is $\xi^{\mathcal{H}} = P^{\mathcal{H}}(\xi^{\mathcal{G}})$.*

3.2.2 VWAP is Not Attainable in the Observed Filtration

Definition 3.7. *A random variable $U_T \in L^2(\mathbb{P})$ is $\mathcal{F}$ attainable (Schweizer [32]) if a unique $\mathcal{F}$ adapted predictable process $\gamma^{U, \mathcal{F}}$ exists such that U_T is replicated by the following Itô integral.*

$$U_T = U_0 + \int_0^T \gamma^{U, \mathcal{F}} dX$$

Lemma 3.8. *VWAP is not $\mathcal{F}$ attainable.*

Proof. By contradiction. Assume VWAP is $\mathcal{F}$ attainable and an attainable $\mathcal{F}$ predictable strategy $\xi^{\mathcal{V}, \mathcal{F}}$ exists, then from eqn 1 and eqn 8 above:

$$\mathcal{V}_T = \int_0^T \xi^{\mathcal{V}, \mathcal{G}} dX = \int_0^T \xi^{\mathcal{V}, \mathcal{F}} dX$$

Hence for all $\mathbb{Q}^{\mathcal{G}} \in \mathcal{M}^e(\mathcal{G})$ the product space isometry implies that:

$$\left\| \int_0^T \xi^{\mathcal{V}, \mathcal{F}} dX - \int_0^T \xi^{\mathcal{V}, \mathcal{G}} dX \right\|_{L^2(\mathbb{Q})} = \|\xi^{\mathcal{V}, \mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}\|_{L^2(X, \mathbb{Q})} = 0$$

The equation above implies that $\xi^{\mathcal{V}, \mathcal{G}} = P^{\mathcal{F}}(\xi^{\mathcal{V}, \mathcal{G}})$. However, $\xi^{\mathcal{V}, \mathcal{G}} = 1 - Y_-$ is not $\mathcal{F}_t$ measurable for $0 \leq t < T$ because, by definition, final cumulative volume V_T (eqn 8) is not $\mathcal{F}_t$ measurable $0 \leq t < T$. Therefore $\xi^{\mathcal{V}, \mathcal{G}} \neq P^{\mathcal{F}}(\xi^{\mathcal{V}, \mathcal{G}})$ and no $\mathcal{F}$ attainable strategy $\xi^{\mathcal{V}, \mathcal{F}}$ exists. □

3.2.3 An L^2 Optimal VWAP Strategy for $(\mathbb{Q}, \mathcal{G})$ Martingale Price

The explicit product space conditional expectation (eqn 10 in lemma 3.5) can be applied to give the $\mathcal{F}$ mean square optimal trading strategy for VWAP under any $\mathbb{Q} \in \mathcal{M}^e(\mathcal{G})$ in the associated Hilbert space $L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$. Noting that the price process X is assumed (assumption 2.1) to be continuous and $\mathcal{F}$ adapted:

$$\xi_t^{\mathcal{V}, \mathcal{F}, \mathbb{Q}} = P^{\mathcal{F}}(\xi_t^{\mathcal{V}, \mathcal{G}}) = \frac{\mathbb{E}^{\mathbb{Q}}[\xi_t^{\mathcal{V}, \mathcal{G}} d\langle X \rangle_t | \mathcal{F}_{t-}]}{\mathbb{E}^{\mathbb{Q}}[d\langle X \rangle_t | \mathcal{F}_{t-}]} = \mathbb{E}^{\mathbb{Q}}[\xi_t^{\mathcal{V}, \mathcal{G}} | \mathcal{F}_{t-}] \quad (11)$$

In particular, if $\mathbb{P} \in \mathcal{M}^e(\mathcal{G})$ and price X is a $(\mathbb{P}, \mathcal{G})$ martingale, the $\mathcal{F}$ L^2 optimal strategy is also the minimum variance strategy³:

$$\xi_t^{\mathcal{V}, \mathcal{F}} = 1 - \frac{V_{t-}}{\mathbb{E}[V_T | \mathcal{F}_{t-}]} \quad (12)$$

Remark 3.9. *If price X is a $(\mathbb{P}, \mathcal{G})$ martingale, then the minimum variance VWAP strategy is the prediction at time $t < T$ of final volume V_T .*

3.2.4 An L^2 Optimal Strategy for Semimartingale X

This section is based on the theory of optimal L^2 hedging for continuous semimartingale processes. The main results of the extensive literature (summarized in Schweizer [33]) are sketched below.

Using the $\mathcal{F}$ VOMM (definition 3.2) density $\tilde{Z}^{\mathcal{F}}$, the process $\hat{Z}_t^{\mathcal{F}}$ can be defined as follows:

$$\hat{Z}_t^{\mathcal{F}} = \mathbb{E}^{\tilde{\mathbb{Q}}} \left[\frac{d\tilde{\mathbb{Q}}^{\mathcal{F}}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] = \frac{\mathbb{E}[(\tilde{Z}^{\mathcal{F}})^2 | \mathcal{F}_t]}{\mathbb{E}[\tilde{Z}^{\mathcal{F}} | \mathcal{F}_t]} \quad (13)$$

There exists an integrand $\zeta^{\mathcal{F}} \in \Theta(\mathcal{F})$ (lemma 2.2 Delbaen and Schachermayer [7]) such that the process $\hat{Z}^{\mathcal{F}}$ can be expressed as the integral:

³Note that the strategy is expressed in terms of ‘VWAP volume yet to be traded’.

$$\hat{Z}_t^{\mathcal{F}} = \hat{Z}_0^{\mathcal{F}} + \int_0^t \zeta^{\mathcal{F}} dX \quad (14)$$

For any random variable $H \in L^2(\mathbb{P})$ the $\tilde{\mathbb{Q}}^{\mathcal{F}}$ martingale $W^{H,\mathcal{F},\tilde{\mathbb{Q}}}$ is defined as the $\tilde{\mathbb{Q}}^{\mathcal{F}}$ conditional expectation of H :

$$W_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} = \mathbb{E}^{\tilde{\mathbb{Q}}} [H \mid \mathcal{F}_t]$$

The Galtchouk [12], Kunita and Watanabe [20] (GKW) decomposition of martingale $W^{H,\mathcal{F},\tilde{\mathbb{Q}}}$ with respect to martingale X under VOMM $\tilde{\mathbb{Q}}^{\mathcal{F}}$ can be formulated:

$$W_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} = \mathbb{E}^{\tilde{\mathbb{Q}}}[H] + \int_0^t \xi^{H,\mathcal{F},\tilde{\mathbb{Q}}} dX + L_t^{H,\mathcal{F},\tilde{\mathbb{Q}}}$$

where $L^{H,\mathcal{F},\tilde{\mathbb{Q}}}$ is a $\tilde{\mathbb{Q}}^{\mathcal{F}}$ square integrable martingale strongly orthogonal to X under $\tilde{\mathbb{Q}}^{\mathcal{F}}$. Finally, the mean square optimal integrand $\xi^{H,\mathcal{F}}$ for H under $\mathbb{P}$ is given by the following feedback equation:

$$\begin{aligned} \xi_t^{H,\mathcal{F}} &= \xi_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} - \frac{\zeta_t^{\mathcal{F}}}{\hat{Z}_t^{\mathcal{F}}} \left(W_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} - \mathbb{E}^{\tilde{\mathbb{Q}}}[H] - \int_0^t \xi^{H,\mathcal{F}} dX \right) \\ &= \xi_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} - \zeta_t^{\mathcal{F}} \left(\frac{W_0^{H,\mathcal{F},\tilde{\mathbb{Q}}} - \mathbb{E}^{\tilde{\mathbb{Q}}}[H]}{\hat{Z}_0^{\mathcal{F}}} - \int_0^t \frac{1}{\hat{Z}^{\mathcal{F}}} dL^{H,\mathcal{F},\tilde{\mathbb{Q}}} \right) \end{aligned} \quad (15)$$

3.2.5 An L^2 Optimal VWAP Strategy for Semimartingale X

The VWAP constant term X_0 (initial stock price) cannot be modified by the VWAP trader (remark 2.4). Therefore it is $\mathcal{V}_T - X_0$ (the Itô integral term in eqn 1) that is optimized below. The $\tilde{\mathbb{Q}}^{\mathcal{F}}$ martingale $W^{H,\mathcal{F},\tilde{\mathbb{Q}}}$ is the conditional expectation of $\mathcal{V}_T - X_0$:

$$W_t^{\nu, \mathcal{F}, \tilde{\mathbb{Q}}} = \mathbb{E}^{\tilde{\mathbb{Q}}} \left[\int_0^t \xi_s^{\nu, \mathcal{G}} dX_s \middle| \mathcal{F}_t \right] = \int_0^t \mathbb{E}^{\tilde{\mathbb{Q}}} [\xi_s^{\nu, \mathcal{G}} | \mathcal{F}_t] dX_s$$

The integrand $\xi^{\nu, \mathcal{F}, \tilde{\mathbb{Q}}}$ of the GKW decomposition of $W_t^{\nu, \mathcal{F}, \tilde{\mathbb{Q}}}$ with respect to martingale X under VOMM $\tilde{\mathbb{Q}}^{\mathcal{F}}$ is:

$$\xi_t^{\nu, \mathcal{F}, \tilde{\mathbb{Q}}} = \frac{d \langle \int_0^\cdot \mathbb{E}^{\tilde{\mathbb{Q}}} [\xi_s^{\nu, \mathcal{G}} | \mathcal{F}_s] dX_s, X \rangle_t}{d \langle X \rangle_t} = \mathbb{E}^{\tilde{\mathbb{Q}}} [\xi_t^{\nu, \mathcal{G}} | \mathcal{F}_t]$$

The orthogonal martingale $L^{\nu, \mathcal{F}, \tilde{\mathbb{Q}}}$ is the $\mathcal{F}$ measurable difference:

$$L_t^{\nu, \mathcal{F}, \tilde{\mathbb{Q}}} = \int_0^t (\mathbb{E}^{\tilde{\mathbb{Q}}} [\xi_s^{\nu, \mathcal{G}} | \mathcal{F}_t] - \mathbb{E}^{\tilde{\mathbb{Q}}} [\xi_s^{\nu, \mathcal{G}} | \mathcal{F}_s]) dX_s$$

Substituting this into equation 15 gives the mean square optimal feedback solution (trading strategy) for VWAP trading with semimartingale X ($\zeta_t^{\mathcal{F}}$ is defined in eqn 14 and $\hat{Z}_t^{\mathcal{F}}$ in eqn 13):

$$\xi_t^{\nu, \mathcal{F}} = \mathbb{E}^{\tilde{\mathbb{Q}}} [\xi_t^{\nu, \mathcal{G}} | \mathcal{F}_t] - \frac{\zeta_t^{\mathcal{F}}}{\hat{Z}_t^{\mathcal{F}}} \int_0^t (\mathbb{E}^{\tilde{\mathbb{Q}}} [\xi_s^{\nu, \mathcal{G}} | \mathcal{F}_t] - \xi_s^{\nu, \mathcal{F}}) dX_s \quad (16)$$

The mean square optimal VWAP trading strategy equation has an intuitive interpretation. The first term of the equation, the GKW decomposition of the martingale $\mathbb{E}^{\tilde{\mathbb{Q}}} [V_T - X_0 | \mathcal{F}_t]$, is the approximation to the $\mathbb{P}$ optimal strategy under the ‘closest’ $\mathcal{F}$ adapted equivalent local martingale measure $\tilde{\mathbb{Q}}^{\mathcal{F}}$. The second term is a feedback error correction term based on updating the accumulated strategy error in the time interval $[0, t]$ using the available information at time t . It is also clear that the prediction $\mathbb{E}^{\tilde{\mathbb{Q}}} [\xi_t^{\nu, \mathcal{G}} | \mathcal{F}_t]$ is central to the ‘practical’ implementation of the optimal strategy, and since cumulative volume V_t is $\mathcal{F}$ adapted, this is equivalent (eqn 8) to the prediction of final volume $\mathbb{E}^{\tilde{\mathbb{Q}}} [V_T | \mathcal{F}_t]$. Under the restrictive assumption that price X and final volume V_T are independent (see section 3.4.2) the optimal solution simplifies to the $\mathbb{P}$ prediction $\mathbb{E} [V_T | \mathcal{F}_t]$. A dynamic prediction model of stock volume has been developed by Białkowski, Darolles and Le Fol [4] and an empirical model of the intraday seasonality of relative volume by McCulloch [21].

3.3 Mean Variance Optimal VWAP Strategies

3.3.1 The Minimum Variance VWAP Strategy

The VOMM density $\tilde{Z}^{\mathcal{F}}$ is closely related to the complement of the projection of the constant 1 onto $G_T(\Theta(\mathcal{F}))$. Define closed subspace $K = \text{span}\{1, G_T(\Theta(\mathcal{F}))\} \subset L^2(\mathbb{P})$. Then the VOMM density $\tilde{Z}^{\mathcal{F}} \in K$ and is the normalized projection of the constant 1 onto $G_T(\Theta(\mathcal{F}))^\perp$.

If the operator π is defined as the projection operator from $L^2(\mathbb{P})$ onto the orthogonal complement $G_T(\Theta(\mathcal{F}))^\perp$ and $\xi^{1,\mathcal{F}} \in \Theta(\mathcal{F})$ is the integrand of the projection of 1 onto $G_T(\Theta(\mathcal{F}))$, then:

$$\tilde{Z}^{\mathcal{F}} = \frac{\pi(1)}{\mathbb{E}[\pi(1)]} = \frac{1}{\mathbb{E}[\pi(1)]} - \int_0^T \frac{\xi^{1,\mathcal{F}}}{\mathbb{E}[\pi(1)]} dX \quad (17)$$

From $\mathbb{E}[\pi(1)] = \mathbb{E}[\pi(1)^2]$ and the definition above:

$$\text{Var}[\tilde{Z}^{\mathcal{F}}] = \frac{1}{\mathbb{E}[\pi(1)]} - 1 \quad (18)$$

The minimum variance ($\lambda \rightarrow \infty$ in eqn 3) VWAP strategy $\xi^{\mathcal{V},\infty,\mathcal{F}}$ (see McCulloch [22], eqn 6) is the sum of the L^2 optimal strategy $\xi^{\mathcal{V},\mathcal{F}}$ and the constant weighted integrand of the projection of 1 onto $G_T(\Theta(\mathcal{F}))$:

$$\xi^{\mathcal{V},\infty,\mathcal{F}} = \xi^{\mathcal{V},\mathcal{F}} - (\text{Var}[\tilde{Z}^{\mathcal{F}}] + 1) \mathbb{E} \left[\int_0^T (\xi^{\mathcal{V},\mathcal{G}} - \xi^{\mathcal{V},\mathcal{F}}) dX \right] \xi^{1,\mathcal{F}} \quad (19)$$

Using the definition of the VOMM $\tilde{Q}^{\mathcal{F}}$ above and the formulation of the minimum variance strategy (eqns 18 and 19) the expectation and variance of the minimum variance VWAP strategy are:

$$\mathbb{E} \left[\int_0^T (\xi^{\mathcal{V},\infty,\mathcal{F}} - \xi^{\mathcal{V},\mathcal{G}}) dX \right] = (\text{Var}[\tilde{Z}^{\mathcal{F}}] + 1) \mathbb{E} \left[\int_0^T (\xi^{\mathcal{V},\mathcal{F}} - \xi^{\mathcal{V},\mathcal{G}}) dX \right] \quad (20)$$

$$\begin{aligned}
\text{Var} \left[\int_0^T (\xi^{\mathcal{V}, \infty, \mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right] &= \mathbb{E} \left[\left(\int_0^T (\xi^{\mathcal{V}, \mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right)^2 \right] \\
&- (\text{Var}[\tilde{Z}^{\mathcal{F}}] + 1) \mathbb{E} \left[\int_0^T (\xi^{\mathcal{V}, \mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right]^2
\end{aligned} \tag{21}$$

3.3.2 The Mean Variance Optimal VWAP Strategy

For a more general and detailed treatment of mean-variance optimal trading strategies as the sum of a minimum variance hedging strategy and an independent ‘directional’ price strategy refer to McCulloch [22].

If the price process is semimartingale under the observed filtration $\mathcal{F}$, then the trader may wish to exploit expected price movement to ‘beat’ VWAP. A trader can exploit expected price movement for the benefit of his client by adopting a VWAP trading strategy that is riskier than the minimum variance strategy in return for a positive expected return. The optimal mean-variance VWAP trading strategy is derived using the Markowitz quadratic mean-variance utility function (eqn 3) with a risk aversion coefficient $0 \leq \lambda < \infty$. The mean-variance optimal strategy $\xi^{\mathcal{V}, \lambda, \mathcal{F}}$ can be formulated [22] as the sum of two distinct trading strategies, a minimum variance VWAP hedging strategy $\lambda \rightarrow \infty$; $\xi^{\mathcal{V}, \infty, \mathcal{F}}$ and a standard ‘directional’ price strategy ξ_0^1 independent of the hedging strategy and market VWAP:

$$\xi^{\mathcal{V}, \lambda, \mathcal{F}} = \xi^{\mathcal{V}, \infty, \mathcal{F}} + \frac{\xi_0^{1, \mathcal{F}}}{\lambda} \tag{22}$$

Where the independent *standard* ($\lambda = 1$) mean-variance optimal price ‘directional’ trade is formulated:

$$\xi_0^{1, \mathcal{F}} = (\text{Var}[\tilde{Z}^{\mathcal{F}}] + 1) \xi^{1, \mathcal{F}} \tag{23}$$

The increases in expected return and variance for an optimal mean variance VWAP trading strategy are proportional to the variance of the density of the $\mathcal{F}$ VOMM:

$$\mathbb{E} \left[\int_0^T (\xi^{\mathcal{V}, \lambda, \mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right] = \mathbb{E} \left[\int_0^T (\xi^{\mathcal{V}, \infty, \mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right] + \frac{1}{\lambda} \text{Var}[\tilde{Z}^{\mathcal{F}}] \quad (24)$$

$$\text{Var} \left[\int_0^T (\xi^{\mathcal{V}, \lambda, \mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right] = \text{Var} \left[\int_0^T (\xi^{\mathcal{V}, \infty, \mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right] + \frac{1}{\lambda^2} \text{Var}[\tilde{Z}^{\mathcal{F}}] \quad (25)$$

3.3.3 Mean Variance Risk Aversion λ and VWAP Trade Size

The greater the proportion of total trading that the VWAP trader controls, the easier it is to trade at the market VWAP price. In the limiting case, the trader controls all traded volume and exactly determines the market VWAP irrespective of trading strategy. Thus VWAP risk is proportional to the traded volume that the VWAP trader does not control.

The market relative volume process Y can be written as a weighted sum of the relative volume of other market participants $\bar{Y}_t = \bar{V}_t/\bar{V}_T$ and the relative volume strategy of the VWAP trader $y_t = v_t/v_T$. The proportion β of the total market volume traded by the VWAP trader is then calculated:

$$\beta = \frac{v_T}{\bar{V}_T + v_T} = \frac{v_T}{V_T} \quad 0 < \beta \leq 1$$

The total relative volume can now be decomposed into the relative volume process of other market participants $\bar{Y}_t$ and the relative volume strategy of the VWAP trader y_t .

$$Y_t = (1 - \beta)\bar{Y}_t + \beta y_t$$

Market VWAP $\mathcal{V}_T$ can be rewritten as:

$$\mathcal{V}_T = X_0 + (1 - \beta) \int_0^T (1 - \bar{Y}_{u-}) dX_u + \beta \int_0^T y_u dX_u$$

Therefore if the VWAP trader final volume v_T is fixed and the final volume of other market participants $\bar{V}_T$ cannot be influenced by the VWAP trader, then the proportion of total volume VWAP traded β cannot be optimized (by increasing β). The mean square error of the VWAP trading strategy is specified:

$$\mathbb{E} \left[\left((1-\beta) \int_0^T (y_u - \bar{Y}_{u-}) dX_u \right)^2 \right] = (1-\beta)^2 \mathbb{E} \left[\left(\int_0^T (\xi^{\mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right)^2 \right]$$

It is also readily shown that variance of the VWAP trading strategy is also scaled by $(1 - \beta)^2$ and expected return by $(1 - \beta)$. Hence, including the VWAP traded volume proportion into the mean variance optimization in equation 3 is equivalent to reducing the risk aversion parameter λ by a factor of $(1 - \beta)$:

$$\max_{\xi^{\mathcal{F}}} \left[\mathbb{E} \left[\int_0^T (\xi^{\mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right] - \frac{\lambda(1 - \beta)}{2} \text{Var} \left[\int_0^T (\xi^{\mathcal{F}} - \xi^{\mathcal{V}, \mathcal{G}}) dX \right] \right]$$

Thus a VWAP trader with a βV_T trade size and risk aversion λ has then same mean variance optimal strategy as a VWAP trader with small ($\beta \ll 1$) trade size with risk aversion parameter $(1 - \beta)\lambda$. It is interesting to note that β is a random variable in filtration $\mathcal{F}$ and the specification of risk aversion is uncertain in the VWAP trader information set.

Remark 3.10.

- (i) The optimal mean square (eqn 16) and minimum variance (eqn 19) VWAP strategies ($\lambda \rightarrow \infty$) are unchanged by VWAP trade size.
- (ii) A VWAP trader with a βV_T trade size has a risk aversion parameter of $(1 - \beta)\lambda$. Where λ is the trader risk aversion parameter for a small VWAP ($\beta \ll 1$) trade size. Thus it is optimal for large $0 < \beta < 1$ mean-variance VWAP traders to increase the size of the price ‘directional’ trade for additional return.

$$\xi^{\mathcal{V},\lambda,\mathcal{F}} = \xi^{\mathcal{V},\infty,\mathcal{F}} + \frac{\xi_0^{1,\mathcal{F}}}{(1-\beta)\lambda}$$

3.4 Simplifying Assumptions For VWAP Strategies

The optimal mean variance VWAP strategies (eqns 16, 19, 22 and 5) allow generality in specifying the VWAP asset price process X and the nature of the dependence between price X and final volume V_T (the relationship between filtrations $\mathcal{F}$ and $\mathcal{G}$) but at the cost of the strategies being formulated at an abstract level. Since VWAP is a ‘practical’ trade, it is worthwhile discussing restrictive assumptions on the price process and the (in)dependence between price and final volume in order to simplify the specification of the optimal VWAP strategies. Finally, as an example, we specify an optimal mean variance VWAP strategy using price modeled as a geometric Brownian motion (GBM) with final volume and price assumed to be independent.

3.4.1 The VOMM and Minimal Martingale Measure Coincide

It is clear from the form of the optimal VWAP strategies that an explicit formulation of the density $\tilde{Z}^{\mathcal{F}}$ of the $\mathcal{F}$ VOMM is required. In practice, the $\mathcal{F}$ VOMM is usually not known and is difficult to formulate (Heath, Platen and Schweizer [14]). However, if the price process satisfies the *structure condition* and the resultant mean-variance tradeoff process is deterministic at time T then $\mathcal{F}$ VOMM coincides with the readily formulated Minimal Martingale measure (Schweizer [31], Schweizer [33] and Föllmer and Schweizer [11]). Briefly, the (continuous) price process satisfies the structure condition if the drift term A of the canonical decomposition under $\mathbb{P}$ is absolutely continuous with respect to the predictable quadratic variation of the martingale M :

$$X_t = X_0 + M_t + \int_0^t \alpha d\langle M \rangle$$

The mean-variance tradeoff process is defined:

$$K_t = \left\langle \int_0^\cdot \alpha dM \right\rangle_t = \int_0^t \alpha^2 d\langle M \rangle = \int_0^t \alpha dA$$

If K_T is deterministic then the Minimal Martingale measure (MMM) density, $\check{Z}^{\mathcal{F}} \in \mathcal{M}^e(\mathcal{F})$, is equivalent to the VOMM density ($\mathcal{E}(\cdot)$ is the stochastic exponential):

$$\check{Z}_t^{\mathcal{F}} = \check{Z}_t^{\mathcal{F}} = \mathcal{E}\left(-\int_0^t \alpha dM\right)$$

Using the VOMM and MMM equivalence, the generalized Bayes formula and lemma 4.7 in Schweizer [33], the abstract form of the L^2 optimal VWAP trading strategy (eqn 16) is simplified to:

$$\xi_t^{\mathcal{V},\mathcal{F}} = \frac{\mathbb{E}[\check{Z}_T^{\mathcal{F}} \xi_t^{\mathcal{V},\mathcal{G}} | \mathcal{F}_t]}{\check{Z}_t^{\mathcal{F}}} + \alpha_t \int_0^t \left(\frac{\mathbb{E}[\check{Z}_T^{\mathcal{F}} \xi_s^{\mathcal{V},\mathcal{G}} | \mathcal{F}_t]}{\check{Z}_s^{\mathcal{F}}} - \xi_s^{\mathcal{V},\mathcal{F}} \right) dX_s \quad (26)$$

We remark that if the VOMM is equivalent to the MMM then explicit mean-variance optimal price ‘directional’ strategies are readily formulated, see McCulloch [22] for details.

$$\xi_0^{1,\mathcal{F}} = \alpha_t \mathcal{E}\left(-\int_0^t \alpha dM\right)$$

3.4.2 The Independence of V_T and X

The assumption of the independence of final volume V_T and price X simplifies the form of the optimal VWAP strategy. Under the independence assumption, the canonical decomposition of $X = M + A$ is unchanged under $\mathcal{G}$ (thm 2 page 364, Protter [27]). Therefore, invoking Girsanov’s theorem, for any $\mathbb{Q}^{\mathcal{F}} \in \mathcal{M}^e(\mathcal{F})$ price X is a $(\mathbb{Q}^{\mathcal{F}}, \mathcal{G})$ martingale and hence $\mathbb{Q}^{\mathcal{F}} \in \mathcal{M}^e(\mathcal{G})$.

However, price and *cumulative* volume V_t are considered to be interrelated in the financial market microstructure literature (Epps [9], Jain and Joh [17] and Karpoff [18]). For example, signed volume is an $\mathcal{F}$ adapted (trader

accessible) process that is the difference between cumulative buyer initiated volume (volume traded at the ask price) and cumulative seller initiated volume (volume traded at the bid price) and is influential on price direction (Chan, Chung and Fong [5]). The dependence of final volume V_T and price X will be weaker than signed volume since final volume does not contain any obvious price directional information. Nonetheless, the assumption that final volume V_T is independent of price X is restrictive.

Under the assumption of independence, $\tilde{Z}^{\mathcal{F}} \in \mathcal{D}^e(\mathcal{G})$ is a $(\mathbb{P}, \mathcal{G})$ martingale and is $\mathcal{F}_t$ adapted:

$$\mathbb{E}^{\tilde{\mathbb{Q}}}[\xi_t^{\nu, \mathcal{G}} | \mathcal{F}_t] = \frac{\mathbb{E}[\xi_t^{\nu, \mathcal{G}} \mathbb{E}[\tilde{Z}^{\mathcal{F}} | \mathcal{G}_t] | \mathcal{F}_t]}{\mathbb{E}[\tilde{Z}^{\mathcal{F}} | \mathcal{F}_t]} = \mathbb{E}[\xi_t^{\nu, \mathcal{G}} | \mathcal{F}_t]$$

Therefore the L^2 optimal VWAP trading strategy (eqn 16) is simplified:

$$\xi_t^{\nu, \mathcal{F}} = \mathbb{E}[\xi_t^{\nu, \mathcal{G}} | \mathcal{F}_t] - \frac{\zeta_t^{\mathcal{F}}}{\hat{Z}_t^{\mathcal{F}}} \int_0^t (\mathbb{E}[\xi_s^{\nu, \mathcal{G}} | \mathcal{F}_t] - \xi_s^{\nu, \mathcal{F}}) dX_s \quad (27)$$

In addition, if X is a $(\tilde{\mathbb{Q}}^{\mathcal{F}}, \mathcal{G})$ martingale then the mean-square and minimum variance solutions coincide $\xi^{\nu, \mathcal{F}} = \xi^{\nu, \infty, \mathcal{F}}$ and the expectation of the minimum variance VWAP strategy (eqn 20) is zero:

$$\mathbb{E}^{\tilde{\mathbb{Q}}} \left[\int_0^T \xi^{\nu, \mathcal{G}} dX \right] = 0$$

The prediction ([4], [21]) of relative volume $\mathbb{E}[\xi_t^{\nu, \mathcal{G}} | \mathcal{F}_t]$ (equivalently final volume, see eqn 8) is central to the successful practical implementation of the VWAP strategy in eqn 27. This also implies that VWAP is a ‘virtuous trade’ that minimizes trading price impact by allocating volume (liquidity demand) during periods of maximum turnover (liquidity supply).

3.4.3 Mean Variance VWAP with GBM

Finally, we conclude this section with an example of mean variance optimal VWAP where the price process is assumed to be a Geometric Brownian Motion (GBM), a continuous semimartingale commonly used to describe the price process on financial markets. The GBM satisfies the structure condition and is usually specified as a stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

where W_t is a standard Brownian motion and μ and σ are constants. Hence, the mean-variance tradeoff process is deterministic and the VOMM and MMM coincide:

$$\alpha_t = \frac{\mu}{\sigma^2 X_t}, \quad K_t = \frac{\mu^2}{\sigma^2} t$$

$$\tilde{Z}_t^{\mathcal{F}} = \check{Z}_t^{\mathcal{F}} = \exp\left(-\frac{\mu}{\sigma} W_t - \frac{1}{2} \frac{\mu^2}{\sigma^2} t\right)$$

If we further assume that final volume V_T and price X are independent, then the minimum variance optimal VWAP trade is formulated:

$$\xi_t^{\mathcal{V}, \infty, \mathcal{F}} = \mathbb{E}[\xi_t^{\mathcal{V}, \mathcal{G}} | \mathcal{F}_t] + \frac{\mu}{\sigma^2 X_t} \int_0^t (\mathbb{E}[\xi_s^{\mathcal{V}, \mathcal{G}} | \mathcal{F}_t] - \xi_s^{\mathcal{V}, \mathcal{F}}) dX_s \quad (28)$$

We emphasize that the key to the practical and efficient implementation of this simplified VWAP algorithm is the prediction of final volume V_T given the available information $\mathcal{F}_t$ at time t :

$$\mathbb{E}[\xi_t^{\mathcal{V}, \mathcal{G}} | \mathcal{F}_t] = 1 - \frac{V_t}{\mathbb{E}[V_T | \mathcal{F}_t]}$$

To complete the mean-variance optimal GBM example, we assume that the VWAP trader is trading βV_T of total market volume in the VWAP trade then the price ‘directional’ trade is formulated:

$$\xi^{\mathcal{V},\lambda,\mathcal{F}} = \xi^{\mathcal{V},\infty,\mathcal{F}} + \frac{\xi_0^{1,\mathcal{F}}}{(1-\beta)\lambda}$$

$$\xi_0^1 = \frac{\mu}{\sigma^2 X_t} \left(\frac{X_t}{X_0} \right)^{-\mu/\sigma^2} \exp \left(\left(\frac{\mu}{\sigma^2} - 1 \right) \frac{\mu t}{2} \right)$$

4 Conclusion

We consider a continuous time model of optimal VWAP trading and derive optimal Markowitz mean-variance VWAP trading strategies. These results generalize Konishi's [19] fixed $\mathcal{F}_0$ adapted solution for a martingale Wiener process price to a continuously dynamic $\mathcal{F}_t$ adapted solution for continuous semimartingale price. In addition, we show that the optimal VWAP trading strategies (eqn 16 and eqn 27) are closely associated to the prediction of relative intraday volume ($\mathbb{E}[Y_t|\mathcal{F}_t]$) and minimize trading price impact by allocating trade volume (liquidity demand) during periods of maximum turnover (liquidity supply). Finally, we show that mean-variance optimal VWAP traders with large VWAP trades will trade larger price 'directional' trades for additional expected return.

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